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How Does Physical Space Influence the Novices' and Experts' Algebraic Reasoning?

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abstract

The embodied mathematics paradigm shows that our mathematical conceptual system is not pure and abstract, but it is grounded in our bodily functioning and experiences. As found by Goldstone & Landy (2007a), symbols placed physically closer together tend to be evaluated as synthetically bound and are solved first, even if in some cases that practice is an error from the perspective of the formal rules of mathematics. Based on this finding, we asked if these results would remain the same if we introduce a new variable — the level of expertise. The results show that experts do not have a better performance than novices, but that the former integrate more the spatial indices in their mathematical reasoning. Results, implications and future research directions are discussed.

Keywords: embodied cognition, embodied mathematics, perceptual spacing, expertise, reasoning

Thinking by using symbols is one of the central mysteries facing the cognitive sciences (Landy & Goldstone, 2007a) and a challenge for so many researchers. Traditionally, symbolic reasoning was thought to have an abstract and arbitrary quality that depends on internal structural rules, which do not relate to explicit external forms (Harnad, 1990; Markman & Dietrich, 2000). Mathematics and especially algebraic reasoning is still considered to be the most paradigmatic case of pure symbolic reasoning and for its successful execution, the use of internally available formal operations was thought to be enough (Piaget, 1958 as cited in Landy & Goldstone, 2007a). Most of today's evidence on the symbolic and mathematical reasoning is no longer in agreement with this assumption of "purity". Although notational mathematics is treated as being an abstract symbol system, these notations are nevertheless visually distinctive forms that occur in particular spatial arrangements and physical contexts. There is some evidence that people are influenced by the particular perceptual form of the representations of abstract entities when performing numerical calculation (Campbell, 1994; Zhang & Wang, 2005; McNeil & Alibali, 2005).

The new approach that contains both — the objectivity of mathematical operations and the importance

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of the way in which the physical space is structured — is called *embodied mathematics.* The theory proposes that the nature of mathematics is revealed directly or indirectly (in most of the cases) from the interaction of the mind with the physical world (Nunez, Edwards, & Mataos, 1999; Lakoff & Nunez, 2000; Tall, 2006). This approach emerges from a more general one—the *embodied cognition approach*—which is increasingly supported by empirical data (Barsalou, 2008; Clark, 2008; Gomila & Calvo, 2008; Riegler, 2002; Wilson, 2002). The main idea of this paradigm is that the emergence of cognition is realized through the interaction of the body with external elements from the environment. Therefore, at every age, reasoning depends also on the body (sensations, morphology, emotions and actions) and the context. Accordingly, intelligent behaviour or test performance does not depend only on abstract cognitive operations, but mainly on the way these operations are implemented in the brain and on the fact that they are situated in a body (with a special morphological structure) and in a physical and social environment (Ionescu, 2011).

Landy and Goldstone (2010) showed that spacing is used as a clue for the operation type. They proved that operands were more likely to be added when widely spaced and to be multiplied when narrowly spaced, supporting the theory that individuals encode the information about operation spacing and use this to select the operation. For example, if the participants were asked to solve the expression "3 + 4×5 =" where the operands that need to be multiplied are closely spaced (the classic practice), results are more accurate than when the expression looked like "3+4 × 5=", where operands that need to be added are closer, in which case the participants are more tempted to do the addition first and then the multiplication. Thus, it is showed that when the spaces between the multiplications are wide and those between additions are narrow, the order of precedence is more likely to change and the participants are prone to wrongly perform the addition first instead of multiplication.

In a series of experiments (Landy & Goldstone, 2007a), the participants were asked to judge if individual simple equations as $a+b*c \;\; +d \;=\; a+c*d \;\; +a$ were mathematically valid. The results show that the physical spacing of formal equation has a large impact on successful evaluation of validity and that the symbols placed physically closer together tend to be syntactically bound.

Our main interest was to see if these results remain the same if we introduce a new variable—*expertise*, namely proficiency in mathematics. So, we are interested in seeing if the participants are influenced in giving the correct answer by the way the mathematical expressions are written as a function of their expertise level. More specifically, we want to verify the assumption (based on embodied cognition approach) that both the novices and the experts are similarly influenced by the way the expressions are written.

Method

This experiment used Landy and Goldstone's (2007a) procedure in which the participants need to judge the validity of 300 mathematical expressions with standard and nonstandard spatial relationship.

Our design was a mixed factorial one (repeated measured design) in which we have manipulated three factors. The first one was *consistency* in which the perceptual grouping was consistent, inconsistent or neutral with respect to the multiplication-before-addition precedence rule. The second one was *sensitivity* referred to whether or not the validity/invalidity would be preserved given precedence of either multiplication-before-addition or addition-before-multiplication; when the equation's validity under the multiplication-before-addition rule differs from that found using the incorrect addition-before-multiplication rule (column 5 from Table 1), that equation is called sensitive). The third factor manipulated was *validity* and there were two possibilities, the equation could be valid or invalid. The *expertise in mathematics* was a nominal variable and has two modalities: experts and novices. We measured the accuracy of responses and reaction times.

Participants

A number of 40 undergraduate students from University of Cluj Napoca, Romania participated as volunteers. Twenty of them were studying Mathematics and Informatics (the group of experts) and twenty were studying Psychology and Social Work (the group of novices).

Stimuli

The task was made in SuperLab 4. The expressions were presented in black text on a white background, using Lucida Grande font, 28 points. Symbols were separated by 1.6 mm, 4.8 mm or 12.7 mm, depending on condition (see the Procedure). Participants used the keyboard to report validity judgements. The keys P and Q were used for invalid and valid expression judgments, respectively.

Procedure

Each participant viewed 240 stimuli and 60 distractors one at a time on the screen of a laptop. Each stimulus expression consisted of two equations (the left side and the right side) separated by an equal sign. Each equation consists of four symbols, connected by three operators (see Table 1). One test stimulus looked like: a * b + c * d = b * a + d *c.

Table 1

Permutations and Mathematical Proprieties of Right-Hand Side Orderings

Permutations	Possible left-hand side	Right-hand side	Validity	Valid if addition is made before multiplication	Sensitivity
a b c d	a + b * c + d	=a + b * c + d	TRUE	TRUE	Insensitive
d c b a	a + b * c + d	=d + c * b + a	TRUE	TRUE	Insensitive
b c a d	a + b * c + d	=b + c * a + d	FALSE	FALSE	Insensitive
c a d b	a + b * c + d	=c + a * d + b	FALSE	FALSE	Insensitive
a c b d	a + b * c + d	=a + c * b + d	TRUE	FALSE	Sensitive
d b c a	a + b * c + d	=d + b * c + a	TRUE	FALSE	Sensitive
c d a b	a + b * c + d	=c + d * a + b	FALSE	TRUE	Sensitive
b a d c	a + b * c + d	=b + a * d + c	FALSE	TRUE	Sensitive

Examples of mathematical expression for all types of consistency are given in the following paragraph.

- Consistent equation: a + b*c + d = d + c*b + a (the spaces between addition were separated by 12.7 mm; the spaces between multiplication were separated by 1.6 mm);
- Neutral equation: a + b * c + d = d + c * b + a the spaces between addition and multiplication were equally separated by 4.7 mm);
- Inconsistent equation: a+b * c+d = d+c * b+a (the spaces between addition were separated by 1.6 mm; the spaces between multiplication were separated by 12.7 mm).

The operand order could differ on right and left hand sides. In order to discourage the participants from using tricks based on symbols constraints used in equations, after every fourth test equation a distractor was presented (an expression with different symbols on each side of the equation, division, subtractions and parentheses: v * [a – (b * c) : d] – f = f – v * [a – (b * c) : d]). Participants had to proceed quickly, without sacrificing accuracy. There was no time restriction; the equation remained on the screen until the participants responded. If the participant's response was correct a check mark was presented as feedback and if the response was wrong an X mark was presented. All the participants received the equations in a completely random order (the same for all the participants). After every trial of 50 stimuli there was a break.

Results

Differences between experts and novices in accuracy responses

Firstly, according to data given by descriptive statistics (Table 2), we can see that experts had a better accuracy on every type of expression.

Table 2

The means and standard deviations from accuracy measures, for all the expressions split by consistency and sensitivity

	Group	Mean	Std. Deviation
Consistent & Sensitive	Novices	.96	.04
	Experts	.99	.01
	Total	.98	.03
Neutral & Sensitive	Novices	.95	.06
	Experts	.97	.03
	Total	.96	.05
Inconsistent & Sensitive	Novices	.82	.21
	Experts	.90	.10
	Total	.86	.16
Consistent & Insensitive	Novices	.97	.02
	Experts	.98	.02
	Total	.97	.02
Neutral & Insensitive	Novices	.96	.05
	Experts	.96	.03
	Total	.96	.04
Inconsistent & Insensitive	Novices	.95	.05
	Experts	.95	.04
	Total	.95	.05

A mixed-group factorial analysis was performed on the accuracy of responses. Significant main effects were found for consistency $F(2, 76) = 25.577$, MSE = .179, $p < .05$ and for sensitivity $F(1, 38) = 8.674$, MSE = .049, $p < .05$. As a between-subjects variable, the expertise had no significant effect [$F(1, 38) = 2.821$, MSE = .037, $p = .10$], therefore we cannot conclude that there are significant differences between the accuracy responses of experts and novices.

As for consistency, there was a difference between the three levels in the performance that experts and novices had. Performing a one-way ANOVA, we found that overall, the experts had a better performance than the novices (see Figure 1), and the only significantly different performance was for the inconsistently spaced expressions $F(1, 38) = 6.243$, MSE = .150, $p < .05$. Concerning the levels of consistency, we did not find significant differences. For neutral expressions $F(1, 38) = 0.45$, MSE = .002, $p = .50$ and for consistent expressions $F(1, 38) = 0.047$, MSE = .000, $p = .82$.

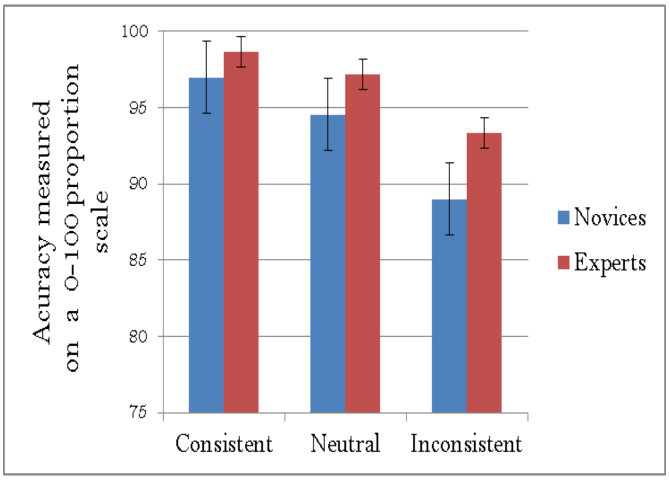

Figure 1. Differences in accuracy (measured on a 0-100 percentage scale) between experts and novices in the consistency modalities. The error bars represent the standard deviations.

Similar to the results presented in Goldstone & Landy (2007), we found that the interaction between sensitivity and consistency is statistically significant, $F(1, 38) = 1.60$, MSE = .001, $p < .05$. For a better view of these interactions see Figure 2. The other interactions were not statistically significant.

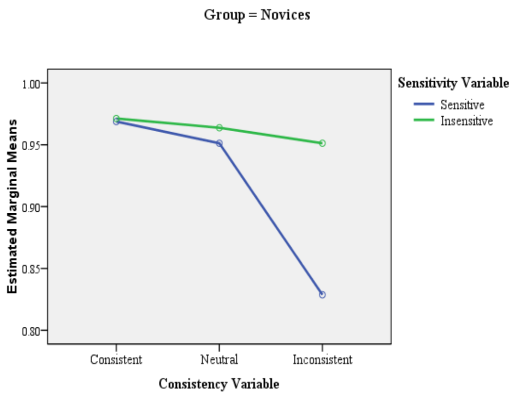

Figure 2. The interactive effect of consistency and sensitivity on the accuracies of novices. The means for accuracy responses were reported.

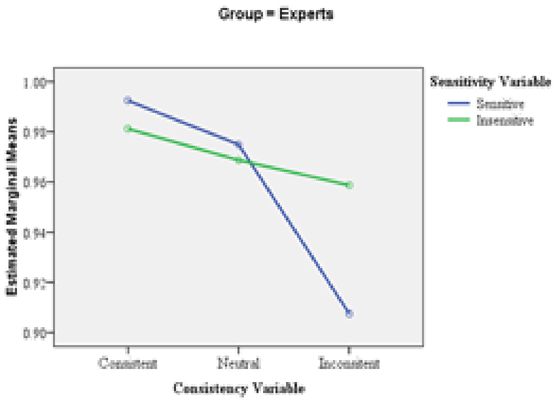

Figure 3. The interactive effect of consistency and sensitivity on the accuracies of experts. The means for accuracy responses were reported.

Figure 2 and 3 shows that the impact which sensitivity had on consistency is not strongly connected to the group involved therefore no other statistics were reported.

Differences between experts and novices in reaction times

As the descriptive statistic shows (Table 2) we can see that on all types of expressions the experts had smaller reaction times than novices.

Table 2

The means and standard deviations for reaction times (measured in milliseconds) on all types of expressions from experiment

	Group	Mean	Std. Deviation
Consistent & Sensitive	Novices	3748.45	923.20
	Experts	2714.48	752.39
	Total	3231.46	982.42
Neutral & Sensitive	Novices	3958.40	983.82
	Experts	2883.84	880.76
	Total	3421.12	1070.30
Inconsistent & Sensitive	Novices	4639.19	1859.02
	Experts	3264.01	912.25
	Total	3951.60	1604.37
Consistent & Insensitive	Novices	3496.04	990.88
	Experts	2720.19	983.83
	Total	3108.12	1050.83
Neutral & Insensitive	Novices	4091.33	1010.29
	Experts	2933.33	833.60
	Total	3512.33	1086.11
Inconsistent & Insensitive	Novices	4359.96	931.66
	Experts	3314.46	918.55
	Total	3837.21	1055.55

A mixed-group factorial analysis was performed to examine the effects of expertise on the reaction times. We found significant main effects only for consistency $F(2, 76) = 50.81$, MSE = 2.1, $p < .05$. For sensitivity $F(1, 38) = 0.77$, MSE = 143139.05, $p = .38$. For the interaction between consistency and sensitivity we did not find either any significant effect $F(2, 37) = 1.44$, MSE = 294619.27, $p = .32$. As a between-subjects variable, the expertise had a significant effect, $F(1, 38) = 13.11$, MSE = 6.66, $p < .05$.

In Figure 4 and 5, we can see what the reaction times were for all kinds of expressions presented, depending on the level of expertise.

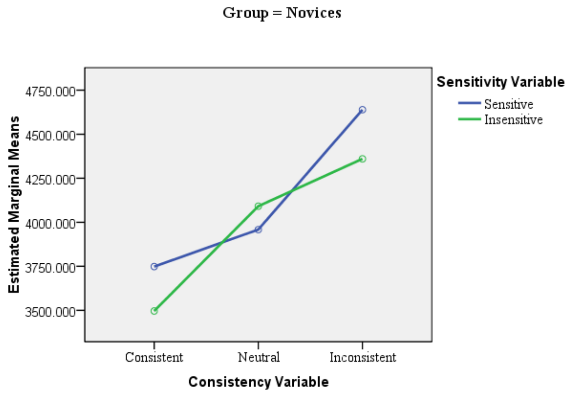

Figure 4. Mean reaction times as a function of sensitivity and consistency for novices.

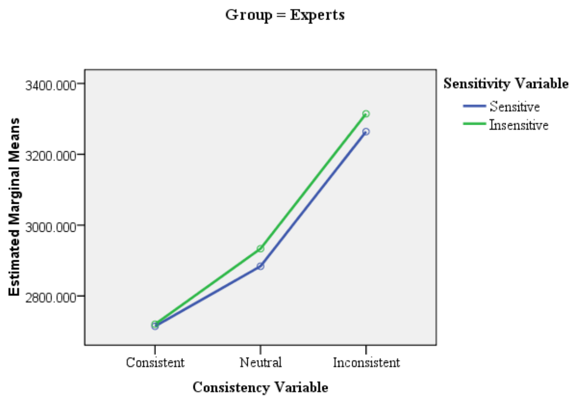

Figure 5. Mean reaction times as a function of sensitivity and consistency for experts.

We compared the performance between novices and experts on every kind of expressions using independent sample T tests. Using the Bonferroni correction we found significant differences at corrected alpha level ($p < .008$) on *sensitive & consistent* trials $t(38) = 3.88$, on *sensitive & neutral* trials $t(38) = 3.63$, on *sensitive & inconsistent* trials $t(38) = 2.97$, on *insensitive & neutral* trials $t(38) = 3.95$, *on insensitive & inconsistent* trials $t(38) = 3.57$, the only not significant result was for *insensitive & consistent* trials $t(38) = 2.48$, $p = .017$.

Discussion

The main goal of our experiment was to test the hypothesis about the perceptual space in mathematical equations having the same influence over the two groups of participants with different levels of expertise in Mathematics. To our knowledge, there are no studies in the area of embodied cognition and embodied mathematics to show whether there is a difference in the way experts and novices perceive space. Our results show that there are some important differences which are briefly discussed below. Hence the results of the study could bring us some valuable information regarding the phenomenon.

If we do not take into account the expertise variable and look at mean accuracy for consistency and sensitivity variable, the results show us that people are influenced by space in their reasoning when judging the validity of the algebraic equations (as found by Goldstone and Landy, 2007a). The results, without considering the expertise variable, can be regarded as a replicate experiment of Landy and Golstone's study. Thus, we found that the participants made more errors in their judgements on those equations that were written in an inconsistent manner than on those equations that were written in a consistent or neutral manner. Moreover, participants were far more affected by spatial consistency when the order of operations judgement affected the correct answer. The replicate experiment of Landy and Goldstone (2007a) brings us additional data to confirm the fact that the spacing in mathematical equations has a large impact on successful evaluations of validity. Symbols placed physically closer together tend to be evaluated as syntactically bound and solved in the first place, even though this procedure may be wrong from the perspective of formal mathematics (Landy & Goldstone, 2007b). Our participants were more familiar with the normative and consistent manner of spacing algebraic equations than with inconsistent ones and the former are better solved

With concern to our hypothesis, we found that there are some important differences between experts and novices when the spacing from equation is consistent, neutral or inconsistent with the precedence order of mathematics and when there is a possibility that the order of operation to be misinterpreted. We found that experts had significant better accuracy responses than novices on inconsistently spaced expressions. The result may suggest that our experts, which may have had more perceptual expertise with the traditional modality of equation spacing, have a better ability to integrate the space in their judgements.

Considering how the sensitivity affects the consistency spatiality in the case of novices, we found that the novices were largely influenced by those expressions that were inconsistent, prominently sensitive. This may suggest that the novices, who may have had less perceptual expertise with the traditional modality of equation spacing, have a worse ability to integrate the space in their judgements.

The analysis of the influences of spacing on reaction times responses gives us more interesting information. Again, as in Landy and Goldstone's experiment (2007a), we did not find significant main effects for consistency and sensitivity and the interaction of these variables. Instead, the expertise level proved to have a significant effect on the speed time in which the expressions were solved. The experts solved all kinds of equations more quickly than novices, except the *insensitive & inconsistent* trials which were solved in the same manner by the two groups. These being said, even if we did not find a significant difference between novices and experts on the accuracy responses, we might say that possibly this is a consequence of the fact that the expression might have been too easy to solve or too familiar. With all these, the analysis of reaction times showed that there are important differences where it is important to look on. The experts had an overall better reaction time on solving the experiment than the novices, and this can give us evidence that there is another perspective on the way the experts and novices perceive space in mathematics, which needs further study.

To conclude, we found that people are influenced in their reasoning by the consistency of space with the precedence order rule. It seems that, regardless of expertise, perceptual grouping influences reasoning. It is possible that experts integrate better the spatial indices in their reasoning or they are better at filtering out the special indices; this could explain why they have better reaction times than novices and why the experts solve the inconsistent trials better than novices.

The difference between experts and novices is of great interest for a number of researchers. Chi, Glaser, and Rees (1982) showed that experts and novices perceive physics problems in different ways. Herbert, Herbert and Larkin (1980) showed that expert and novices build different representations for different kind of problems, even when these problems are very simple. From the perspective of embodied cognition approach, the results of our study bring additional evidence supporting the paradigm that all people are influenced in their reasoning by non-formal clues from the environment. Even for experts, what truly matters is not only the abstract rules, but also the perceptual stimuli. Moreover, these clues seem to be integrated in reasoning and foster the accuracy (or at least the reaction times) of formal judgements.

The main implications of this study are for education. The modality of presenting and the physical format used in teaching may have a great importance for the manner in which informational content is understood and memorised. Understanding the way in which people mentally represent mathematics is important both for building more efficient teaching methods of mathematic and for improving the modality in which mathematical notations are written. Mathematical and abstract systems that align the non-formal indices with formal regularities are easier to be understood and used than those systems that do not follow the consistencies (Figure 1).

We need to consider the limits of our study. Firstly, for a better sampling we should have given the participants some mathematical test to see exactly their level in mathematics or take as experts only those persons that have exceptional results in mathematics. Another limit can be the fact that the equations were very simple (even for novices) and maybe an average difficulty would have kept more interest for the probe for both groups of participants.

A lot of participants claimed the fact that the task was too long, and their interest in paying attention until the end has diminished. The boredom appeared while the experiment was on track which may have led to a reduced involvement of participants in the given task (however, no participant stopped performing the experiment). We could eventually verify whether this is a pattern or if the sensitivity of our data could have been increased by discarding the last blocks of expression. With all this, we to think that including all the expressions can give a more ecologic perspective of the study (it can show what happens in situations when someone has to do something that is repetitive, considered too easy and the resources are not distributed accordingly—respectively that other elements than rules determine what we are looking on, in these case—the space). Another problematic aspect can be the fact that the presentation order was the same for all the participants, which needs to be taken into account by future studies.

It would be interesting to go further with the study of the influence of space in children, at various ages. These could bring us a more powerful explanation about what happens with the influence of different spatial arrangements on algebraic reasoning across the development (it is wider, smaller or the same).

Another way to see whether there are differences between experts and novices in the way they use the space in their reasoning is to examine novel mathematical languages that are learned in various situations (Landy & Goldstone, 2007c). The research based on these studies has many implications for cognitive scientists interested in abstract pattern learning.

Taking into consideration the great implication (for fundamental research and education) of data obtained from studies which investigate the embodied cognition approach, more studies are necessary, with more refined experimental designs.

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